ON SOLVABILITY OF NONLINEAR EQUATIONS OF REISSNER FOR NONSHALLOW SYMMETRICALLY LOADED SHELLS OF REVOLUTION

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In 1950 Reissner [1] derived equations for finite symmetrical deformation of thin shells of revolution, the surface of which is given by parametric Eqs. $r = r(\xi)$ and $z = z(\xi)$ ξ is the parameter), without assumption about any degree of smallness of the angle of rotation of the shell element as a result of the deformation. In the case of a shallow shell and the assumption of smallness of angles of rotation, from these equations the well known equations of the shallow theory may be obtained. At the present moment the study of the latter has made a certain amount of progress, however the necessity for studying more exact equations of nonlinear theory of shells becomes more and more apparent.

This work is devoted to the proof of the existence theorem of solutions of Reissner's equations for various boundary conditions but with some limitations placed on the class of shells to be studied. Namely, we shall assume that for $0 < a \leqslant \xi \leqslant b < \infty$, i.e. in the range of variation of parameter ξ the following conditions apply

$$0 < l_1 \leqslant \frac{r}{\alpha} \leqslant l_2 < \infty, \qquad \alpha^2(\xi) = \left(\frac{dr}{d\xi}\right)^2 + \left(\frac{dz}{d\xi}\right)^2$$

This includes a rather broad class of shells of revolution such as cylindrical, toroidal, spherical with a cutout pole, ring-shaped, plate-shaped etc. However, this conditions excludes some important types of shells such as spherical dome and circular plate. We shall assume that this limitation is not related to the substance of the matter and hope to remove it in the future.

Existence theorems in the nonlinear theory of shallow shells were obtained in papers of Vorovich [2 and 3] for a variety of boundary conditions. The problem to be examined below in the theory of nonshallow shells is interesting from the point of view of methodology because together with nonlinear differential equations of equilibrium and simultaneous shell deformation, nonlinear boundary conditions also apply. For the proof of the existence theorem here the method of paper [3] is applied.

The following system of nonlinear differential equation is examined.

$$\frac{d}{d\xi}\frac{r}{\alpha}\frac{d\beta}{d\xi} = \frac{\alpha}{r}\cos(\Phi_0 - \beta)\left[\sin\Phi_0 - \sin(\Phi_0 - \beta)\right] + \nu\frac{d\Phi_0}{d\xi}\left[\cos(\Phi_0 - \beta) - \cos\Phi_0\right] -$$

$$- \frac{\alpha}{D}\left[\Psi\sin(\Phi_0 - \beta) - T\cos(\Phi_0 - \beta)\right] \equiv F_1(\beta, \Psi)$$

$$\frac{d}{d\xi}\frac{r}{\alpha}\frac{d\Psi}{d\xi} = \left[\frac{\alpha}{r}\cos^2(\Phi_0 - \beta) - \nu\frac{d(\Phi_0 - \beta)}{d\xi}\sin(\Phi_0 - \beta)\right]\Psi + \qquad (1)$$

$$+ \alpha C\left[\cos(\Phi_0 - \beta) - \cos\Phi_0\right] + \nu\sin(\Phi_0 - \beta)\frac{dT}{d\xi} +$$

$$+\left[\frac{\alpha}{r}\cos(\Phi_0-\beta)\sin(\Phi_0-\beta)+\nu\frac{d(\Phi_0-\beta)}{d\xi}\cos(\Phi_0-\beta)\right]T-$$
$$-\left[\frac{d}{d\xi}(r^2p)+\nu r\alpha p\cos(\Phi_0-\beta)\right]\equiv F_2(\beta,\Psi)$$

$$T=-\int r\alpha q d\xi,\qquad C=Eh,\qquad D=\frac{Eh^3}{12(1-\nu^2)}$$
$$\Phi=\Phi_0-\beta,\qquad \alpha^2(\xi)=\left(\frac{dr}{d\xi}\right)^2+\left(\frac{dz}{d\xi}\right)^2,\qquad 0<\nu<0.5$$

with the boundary conditions

$$\beta=0,\qquad \Psi=0\qquad \text{при }\xi=a$$
$$\frac{r}{\alpha}\frac{d\beta}{d\xi}=\nu[\sin(\beta-\Phi_0)-\sin\Phi_0]\equiv f_1(\beta)\ \text{ for }\ \xi=b \tag{2}$$
$$\frac{r}{\alpha}\frac{d\Psi}{d\xi}=\nu\Psi\cos(\Phi_0-\beta)+T\sin(\beta-\Phi_0)-r^2p\equiv f_2(\beta,\Psi)$$

Problem (1), (2) is the system of equations of Reissner for symmetrical deformation of a thin shell of revolution with constant thickness. The surface of the shell is given by parametric Eq. $r=r(\xi)$ and $z=z(\xi)$. Here $\Phi_0(\xi)$ is the angle which the element of the shell forms at the point corresponding to ξ before deformation with the axis of the abscissa; $\Phi(\xi)$ is the angle after deformation, Ψ is the horizontal component of stress; $T\equiv(rV)$ is the vertical component of stress, E is Young's modulus, ν is Poissan's ratio, $p\equiv p_h$ and $q\equiv p_v$ are the horizontal and vertical components of loading which depend on the intensity of loading $\rho(\xi)$ and the angle $\beta(\xi)$. For example, in the case of the spherical dome under hydrostatic pressure of constant intensity ρ, we have

$$r(\xi)=R\sin\xi,\qquad z(\xi)=-R\cos\xi,\qquad \Phi_0=\xi,\qquad p=-\rho\sin(\xi-\beta),\qquad q=\rho\cos(\xi-\beta)$$

while in the case of a cylindrical shell it is necessary to write $r=R$, $z=R\xi$ and $\Phi_0=\frac{1}{2}\pi$ where R is the radius of the base of the cylinder.

Boundary conditions (2) are selected for the sake of definiteness. The first of these for $\xi=a$ indicades that the corresponding edge of the shell is rigidly clamped and is free of stresses. The second condition for $\xi=b$ describes fixed hinge-type attachment of the shell along the edge.

Let us introduce Banach spaces of functions

1) Spaces C_k of continuous functions, definite in the interval $[a,b]$, having derivatives to k-th order included, with the norm

$$\|w\|_{C_k}=\sum_{l=0}^{k}\max_{a\leqslant\xi\leqslant b}\left|\frac{d^lw}{d\xi^l}\right|$$

2) Space H_1 of pair of functions $x\equiv(x_1,x_2)$, where $x_1\in C_1$, and $x_2\in C_1$ with the norm

$$\|x\|_{H1}=\|x_1\|_{C1}+\|x_2\|_{C1}$$

3) Space H_2 of pair of functions $y\equiv(y_1,y_2)$, where $y_1\in C_2$, and $y_2\in C_2$, with the norm

$$\|y\|_{H2}=\|y_1\|_{C2}+\|y_2\|_{C2}$$

4) Space H_3 of pair functions $\sigma\equiv(\sigma_1,\sigma_2)$, formed by closure with respect to the norm of the set of smooth functions vanishing at $\xi=a$:

$$\|\sigma\|_{H_3}^2=\int_a^b\left[\left|\frac{d\sigma_1}{d\xi}\right|^2+\left|\frac{d\sigma_2}{d\xi}\right|^2\right]d\xi$$

5) Space H_4 of pair of functions $z\equiv(z_1,z_2)$, formed by closure with respect to the norm

of the set of smooth functions vanishing for $\xi = a$:

$$\| z \|_{H_1} = \left[\int_a^b \left(\left| \frac{d^2 z_1}{d\xi^2} \right|^2 + \left| \frac{d^2 z_2}{d\xi^2} \right|^2 \right) d\xi \right]^{1/2} + \left| \frac{dz_1(b)}{d\xi} \right| + \left| \frac{dz_2(b)}{d\xi} \right|$$

Theorem 1. Let

$$\alpha^2 q \in C_0,\ \ \alpha^2 p' \in C_0,\ \ \alpha' \in C_0,\ \ 0 < l_1 \leqslant r/\alpha \leqslant l_2 < \infty,\ \ a \leqslant \xi \leqslant b \tag{3}$$

Then the boundary value problem (1), (2) has at least one solution $u \equiv (\beta, \Psi)$, components of which are elements of space C^2.

For the proof of the theorems we shall apply the principle of Lerei and Shauder [4 and 5] on the existence of fixed points of completely continuous transformations.

We shall examine the family of operators depending on parameter $\lambda \in [0, 1]$

$$\frac{d}{d\xi}\frac{r}{\alpha}\frac{d\beta}{d\xi} = \lambda f_1(\beta, \Psi), \qquad \frac{d}{d\xi}\frac{r}{\alpha}\frac{d\Psi}{d\xi} = \lambda F_2(\beta, \Psi) \tag{4}$$

with boundary conditions

$$\beta(a) = 0, \qquad \Psi(a) = 0 \tag{5}$$

$$\frac{r}{\alpha}\frac{d\beta}{d\xi} = \lambda f_1(\beta), \qquad \frac{r}{\alpha}\frac{d\Psi}{d\xi} = \lambda f_2(\beta, \Psi) \quad \text{for} \quad \xi = b$$

For $\lambda = 1$ problem (4), (5) transforms into (1), (2), and for $\lambda = 0$ it has a unique solution $\beta = \Psi = 0$.

The following linear problem is compared with the nonlinear problem (4), (5)

$$\frac{d}{d\xi}\frac{r}{\alpha}\frac{d\beta}{d\xi} = \lambda F_1(\varphi, \psi), \qquad \frac{d}{d\xi}\frac{r}{\alpha}\frac{d\Psi}{d\xi} = \lambda F_2(\varphi, \psi) \tag{6}$$

with boundary conditions

$$\beta(a) = \Psi(a) = 0 \tag{7}$$

$$\frac{r}{\alpha}\frac{d\beta}{d\xi} = \lambda f_1(\varphi), \qquad \frac{r}{\alpha}\frac{d\Psi}{d\xi} = \lambda f_2(\varphi, \psi) \quad \text{for} \quad \xi = b$$

which consist of determining functions β, Ψ with respect to known functions φ, ψ. The linear problem (6), (7) is solvable and the uniqueness theorem is applicable to it. Therefore the pair of functions $u \equiv (\beta, \Psi)$ is uniquely determined with respect to $v \equiv (\varphi, \psi)$. This correspondence determines the nonlinear operator

$$u = L(v, \lambda) \tag{8}$$

which is comparable to any λ in $[0.1]$ and $v(\xi)$ from space H_1 of solution of problem (5), (6). Fixed points of transformation $L(u, \lambda)$ will be solutions of problem (4), (5) and vice versa.

In this manner problem (4), (5) is reduced to solution of the problem

$$u = L(u, \lambda) \tag{9}$$

Lemma. For the solution of problem (4), (5) the following estimates are applicable, which are uniform with respect to $\lambda \in [0, 1]$:

$$\int_a^b \frac{r}{\alpha}\left(\frac{d\Psi}{d\xi}\right)^2 d\xi \leqslant m, \quad \max_{a \leqslant \xi \leqslant b} |\Psi| \leqslant m, \quad \int_a^b \frac{r}{\alpha}\left(\frac{d\beta}{d\xi}\right)^2 d\xi \leqslant m \tag{10}$$

where the constant m depends on l_1, $\|\alpha^2 p\|_{L_2}$ and $\|\alpha^2 q\|_{L_1}$.

For the purpose of proof we multiply the second Eq. of system (4) by Ψ and integrate from a to b. Integrating by parts with consideration of boundary conditions (5), we obtain

$$\int_a^b \frac{r}{\alpha}\left(\frac{d\Psi}{d\xi}\right)^2 d\xi + \lambda\int_a^b \frac{\alpha}{r}\Psi^2\cos^2\Phi d\xi - 2\lambda\nu\int_a^b \Psi\frac{d\Psi}{d\xi}\cos\Phi d\xi =$$

$$= \lambda C\int_a^b \alpha(\cos\Phi_0 - \cos\Phi)\Psi d\xi - \frac{\lambda}{2}\int_a^b \frac{\alpha}{r}T\Psi\sin 2\Phi d\xi +$$

$$+ \lambda\nu\int_a^b T\frac{d\Psi}{d\xi}\sin\Phi d\xi - \lambda\int_a^b r^2 p\frac{d\Psi}{d\xi}d\xi + \lambda\nu\int_a^b rp\Psi\cos\Phi d\xi \quad (\Phi = \Phi_0 - \beta)$$

Utilizing the following inequality which is applicable by virtue of (3):

$$|\Psi(\xi)| < \left(\frac{b-a}{l_1}\int_a^b \frac{r}{\alpha}\left(\frac{d\Psi}{d\xi}\right)^2 d\xi\right)^{1/2} \tag{11}$$

and also the inequality of Buniakowski we derive from here

$$\int_a^b \frac{r}{\alpha}\left(\frac{d\Psi}{d\xi}\right)^2 d\xi + \lambda\int_a^b \frac{\alpha}{r}\Psi^2\cos^2\Phi d\xi \leqslant \lambda m_2\left[\left(\int_a^b \frac{r}{\alpha}\left(\frac{d\Psi}{d\xi}\right)^2 d\xi\right)^{1/2} +\right.$$

$$\left.+\left(\int_a^b \frac{\alpha}{r}\Psi^2\cos^2\Phi d\xi\right)^{1/2}\right]\left[\left(\int_a^b \frac{\alpha}{r}T^2 d\xi\right)^{1/2} + \left(\int_a^b \alpha r^3 p^2 d\xi\right)^{1/2} + m_1\right] \tag{12}$$

where m_2 is some constant.

Taking into account the expression for T given in (1), we derive from (12) the first estimate (10) with the aid of (3). The second estimate is obtained by applying (11).

To obtain the third estimate in (10), we multiply the first Eq. of system (4) by β and integrate from a to b. We obtain

$$\int_a^b \frac{r}{\alpha}\left(\frac{d\beta}{d\xi}\right)^2 d\xi = \lambda\int_a^b \frac{\alpha}{r}\cos\Phi[\sin\Phi_0 - \sin\Phi]\beta d\xi +$$

$$+ \lambda\int_a^b\left\{\nu\frac{d\Phi_0}{d\xi}[\cos\Phi_0 - \cos\Phi] - \frac{\alpha}{D}[\Psi\sin\Phi - T\cos\Phi]\right\}\beta d\xi \quad (\Phi = \Phi_0 - \beta)$$

From here, applying (3) to the left part and utilizing the inequality of Buniakowski we obtain the desired estimate. Lemma is proved.

Now we derive the bounds for β and Ψ in the norm C_1. For this purpose first of all we change from system (4), (5) to the equivalent system of integral equations.

For example for the first Eq. we obtain

$$\frac{d\beta}{d\xi} = \frac{\lambda\alpha}{r}\left[\int_a^\xi F_1(t)\,dt - \int_a^b F_1(t)\,dt + f_1(b)\right] \tag{13}$$

(the second Eq. is the same, but with a substitution of F_1 by F_2 and f_1 by f_2).

From explicit expressions F_1, F_2, f_1 and f_2 utilizing estimate (10) we derive

$$\|F_1\|_{C_0} < m_3, \quad \|F_2\|_{L_2} < m_3, \quad \|f_1\|_{C_0} < m_3, \quad \|f_2\|_{C_0} < m_3 \tag{14}$$

for the condition that $p(b) < \infty$ and bounded norms

$$\|\alpha^3 p\|_{L_2}, \qquad \|\alpha^2 p'\|_{L_2}, \qquad \|\alpha^2 q\|_{L_2}$$

By virtue of (14) we have from (13)

$$\max_{a<\xi<b}\left|\frac{d\beta}{d\xi}\right| \leqslant m_4, \qquad \max_{a\leqslant\xi<b}\left|\frac{d\Psi}{d\xi}\right| \leqslant m_4 \tag{15}$$

Finally we shall demonstrate that when the conditions of the theorem are fulfilled the following bounds apply

$$\max_{a<\xi<b}\left|\frac{d^2\beta}{d\xi^2}\right| \leqslant m_5, \qquad \max_{a<\xi<b}\left|\frac{d^2\Psi}{d\xi^2}\right| \leqslant m_5 \tag{16}$$

For this purpose the term with the first derivative in Eqs. (4) is transferred to the right side. Applying (10), (15) and conditions of the Theorem with regard to p and q we evaluate the right-hand parts and also f_1 and f_2 and derive (16).

Let us return to the operator Eq. (9). Let us show that L is a completely continuous operator in the space H_1. For this purpose let us examine $F_1(\varphi, \psi)$, $F_2(\varphi, \psi)$, $f_1(\varphi)$ and $f_2(\varphi, \psi)$. We obtain that F_1, F_2 are elements of space C_0, while f_1 and f_2 are elements of space C_1. Then solutions of problem (6), (7) will belong to space h_2 and estimates (16) will be applicable to them.

Since the set of functions bounded in the norm C_2 is compact in space C_1 we have that any set bounded in H_1 is transformed by operator L into a compact one. The complete continuity of operator L follows from this. It is also simple to establish the gradual continuity of transformation $L(u, \lambda)$ with respect to λ.

Thus all conditions of the Lerei-Shauder principle are satisfied and consequently Eq. (8) has at least one solution in H_1. In fact this solution will be smoother by virtue of (16). Therefore solution of problem (1), (2) will belong to space H_2, and the Theorem is proved.

Theorem 2. Let

$$\alpha^3 q \in L_2, \quad \alpha^2 p \in L_2, \quad \alpha^2 p' \in L_2, \quad \alpha' \in L_2, \quad \alpha^2 p\,|_{\xi=b} < \infty$$

and condition (3) is also applicable. Then the boundary value problem (1), (2) has at least one solution $u \equiv (\beta, \Psi)$, which is an element of space H_4.

The proof is the same as in Theorem 1, it is only necessary to replace space H_1 by H_3 and H_2 by H_4.

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